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Remarks on the functional integration quantization of gravitation*

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Feynman quantization [1] applied to field theory [2, 3] is not entirely satisfactory with regard to the rigorous definition of the functional integrals and the handling of fermion fields. This note concerns the still more uncertain use of the Feynman method for the quantization of generally covariant systems. More specifically, we shall give special attention to Feynman quantization of general relativity.

In order to quantize according to Feynman, it is necessary to choose field variables for the functional integration and to assign an appropriate integration weighting. Misner [4] has approached this problem in a very general manner by considering a transitive group of permutations which operate on the field variables. The integration weighting is defined by requiring it to be invariant under the permutation group. This prescription is apparently dependent on the choice of the permutation group. Furthermore, the appropriate domain for the functional integration is not always obvious.[1]

Now we shall confine our remarks to the quantization of general relativity. The manifestly covariant Lagrangian density for general relativity is $\mathfrak{L} = R\sqrt{g}$. Since our argument will concern the local properties of the weighting, it is justifiable to neglect a complete divergence and write $\mathfrak{L}$ in the well-known form

$$\mathfrak{L} = \sqrt{g}\, G^{\mu\nu\alpha\beta\varkappa\lambda}\, g_{\mu\nu,\varkappa}\, g_{\alpha\beta,\lambda}, \tag{1}$$

where $g = -\operatorname{Det} g_{\mu\nu}$ and $G^{\mu\nu\alpha\beta\varkappa\lambda}$ is a sum of terms, each of which is trilinear in the $g^{\mu\nu}$. Equation (1) is easily obtained by expressing the $\Gamma^{\alpha}{}_{\beta\gamma}$ in the formula

$$\mathfrak{L} = \sqrt{g}\, g^{\mu\nu}\left(\Gamma^{\beta}_{\alpha\nu}\Gamma^{\alpha}_{\mu\beta} - \Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{\mu\nu}\right) \tag{2}$$

(see e.g. Møller [5] p. 335) in terms of the $g_{\mu\nu}$ and their derivatives.

Notice that if $G^{\mu\nu\alpha\beta\varkappa\lambda}$ had been independent of $g_{\mu\nu}$, $\mathfrak{L}$ in (1) would have had exactly the form of the kinetic energy in a linear theory, and we would have no doubt about the correct measure to use [1]. Written formally, the weighting would be $\Pi \mathrm{const} \cdot dg_{\mu\nu}$, where the Π-product is taken over all ten components of the metric tensor $g_{\mu\nu}$ and over all space-time points. This simple fact actually dictates the weighting to be used in the construction of the Feynman functional integrals. Consider the Feynman integral

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[1] Misner restricts the metric variables to the ordinary signature (+ + + −). The natural choice is then the group of all linear transformations independently performed on the metric for each space-time point.

$$\int_{\Omega_x} e^{i \int_{\Omega_x} \mathcal{L} d^4x} \delta g_{\mu\nu}, \tag{3}$$

where Ω_x is a four-volume which contains the point x. If we take Ω_x sufficiently small, $g_{\mu\nu}$ assumes the same value at all points on the boundary of Ω_x for all significant field histories. This follows from the fact that all significant field histories are continuous [1]. Hence, with an error which goes to zero as Ω_x converges to the point x, we can replace $g_{\mu\nu}$ through Ω_x with the values $g_{\mu\nu}(x)$. Then $G^{\mu\nu\alpha\beta\varkappa\lambda}$ through Ω_x assumes the values $G^{\mu\nu\alpha\beta\varkappa\lambda}(x)$.

The integration over the variables inside Ω_x must now be performed with the weighting

$$\delta g_{\mu\nu} = \prod C(g_{\mu\nu}(x))\, d g_{\mu\nu} \quad (\text{for } x \in \Omega_x) \tag{4}$$

with the Π-product taken over all ten components of the metric tensor and all points in Ω_x. The function C can be normalized by putting

$$C(\mathring{g}_{\mu\nu}) = 1 \tag{5}$$

where $\mathring{g}_{\mu\nu}$ is the Lorentz metric. If we perform a linear coordinate transformation such that

$$g_{\mu\nu}(x) \rightarrow \mathring{g}_{\mu\nu} \tag{6}$$

we find that

$$C(g_{\mu\nu}(x)) = \mathrm{Det}\left(\frac{\partial \mathring{g}_{(\mu\nu)}}{\partial g_{(\alpha\beta)}(x)}\right). \tag{7}$$

The indices μ, ν (and α, β) are here surrounded by brackets to indicate that they must be understood in terms of one index which ranges from one to ten. Hence the determinant in (7) is a ten by ten determinant.

The form for the integration weighting (4) is obviously valid for all x, so we have

$$\delta g_{\mu\nu} = \Pi\, C(g_{\mu\nu})\, dg_{\mu\nu} \quad (\text{for all } x) \tag{8}$$

in which equation

$$C(g_{\mu\nu}) = \mathrm{Det}\left(\frac{\partial \mathring{g}_{(\mu\nu)}}{\partial g_{(\alpha\beta)}}\right). \tag{9}$$

A simple group theoretical argument (see appendix) shows that

$$C(g_{\mu\nu}) = |C_\mu{}^\alpha|^\beta \tag{10}$$

where β is a certain set of numbers and the $C_\mu{}^\alpha$ fulfill[1]

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[1] We exclude reflections, so that $|C_\mu{}^\alpha| > 0$.

$$\mathring{g}_{\mu\nu} = C_\mu^{\ \alpha} C_\nu^{\ \beta} g_{\alpha\beta} \tag{11}$$

(10) and (11) give

$$C(g_{\mu\nu}) = \left(\frac{|\mathring{g}_{\mu\nu}|}{|g_{\mu\nu}|}\right)^{\beta/2}. \tag{12}$$

Observe that C is homogeneous of degree -10 in $g_{\mu\nu}$, while $g = -|g_{\mu\nu}|$ is homogeneous of degree 4 in $g_{\mu\nu}$. Thus we finally obtain

$$C(g_{\mu\nu}) = (g/\mathring{g})^{-5/2}. \tag{13}$$

Equations (8) and (13) constitute Misners result [4]. It is obvious (as Misner has pointed out) that the weighting (8), (13) is invariant with respect to coordinate transformations.

It is easy to see that the result (13) can be written in the following form (neglecting constant factors):

$$C(g_{\mu\nu}) = \sqrt{\mathrm{Det}\,(g_{\varkappa\lambda}\, G^{(\mu\nu)(\alpha\beta)\varkappa\lambda})}. \tag{14}$$

The expression (14) is an invariant generalisation of the weighting prescription in particle mechanics [6].

It is very easy to show that one need not worry about the nonphysical gauge variables when using the functional integration formulation of quantization [7]. Let S be a gauge-invariant[1] Feynman exponent (i times the action) and F a functional of the fields f to be quantized. We require F to be gauge invariant. From all the field histories of f let us select those that are close to a certain gauge, that is, we take out a "slice" of histories with the following property: A certain subgroup of the gauge transformation group applied to the slice generates all admissible histories once and only once. The more the subgroup contains, the "thinner" is the slice. Such a slice is expected to exist in any theory where dynamical variables can be extracted.

Performing the functional integration over the slice only, we have

$$\frac{\int_1 F e^S \delta f}{\int_1 e^S \delta f}. \tag{15}$$

Due to the required invariances

$$\int_1 F e^S \delta f = \int_\nu F e^S \delta f \tag{16}$$

where ν denotes integration over a slice obtained from the original slice with one of the transformations in the subgroup. Similarly we find

$$\int_1 e^S \delta f = \int_\nu e^S \delta f \tag{17}$$

and thus we obtain

$$\frac{\int_1 F e^S \delta f}{\int_1 e^S \delta f} = \frac{\sum_\nu \int_\nu F e^S \delta f}{\sum_\nu \int_\nu e^S \delta f} = \frac{\int F e^S \delta f}{\int e^S \delta f}. \tag{18}$$

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[1] Gauge invariance here is meant in the wider sense which includes general coordinate transformations.

In the last expression the integrations are performed over all admissible f. Here we have assumed that δf is invariant.

In certain theories (e.g. electrodynamics) it is possible to get stronger results [7]. Assume that for a certain choice of slicing, the Feynman exponent can be split into two parts

$$S = S_O + S_G \tag{19}$$

where S_G depends only on the gauge (that is, in which slice a certain history appears, but not where it is in the slice).[1] In this case

$$\frac{\int\limits_1 F e^S \delta f}{\int\limits_1 e^S \delta f} = \frac{\sum\limits_\nu e^{-S_{G\nu}} \int\limits_1 F e^S \delta f}{\sum\limits_\nu e^{-S_{G\nu}} \int\limits_1 e^S \delta f} = \frac{\sum\limits_\nu \int\limits_\nu F e^{S_0} \delta f}{\sum\limits_\nu \int\limits_\nu e^{S_0} \delta f} = \frac{\int F e^{S_0} \delta f}{\int e^{S_0} \delta f} \tag{20}$$

(20) shows that, if we want, we can integrate over all histories, using the part of the Feynman exponent respresented by S_O. This is usually done in electrodynamics. It is important to remember that F must be gauge invariant. In electrodynamics this follows automatically excluding longitudinal and scalar photons from the initial and final states.

Let us briefly return to the problem of the domain for the functional integration. Evidently there is an infinity of different choices. As an example, consider the method of quantizing relativity with the Palatini Lagrangian [8]. In this method the metric variables and the affinity variables are treated as independent. This means that we are integrating over a domain which is much larger than the domain in the Misner formulation. It is not unlikely that the two possibilities will give different results outside the classical limit, and it seems impossible *a priori* to know which one is correct.

So far no one has succeeded in performing the explicit integrations for general relativity. The advantage of the Feynman method lies in the flexibility of the formalism. It is often possible to prove difficult theorems with relatively little effort.

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[1] In the gauge invariant Feynman exponent for electrodynamics there is a term proportional to

$$\int d^4 x\, \partial^\nu A_\nu \cdot \partial^\mu A_\mu$$

that can be used as S_G.

APPENDIX

We have asserted the relation

$$C = |C_\mu{}^\alpha|^\beta$$

where β is a certain set of complex numbers which are independent of $|C_\mu{}^\alpha|$. This can be shown in the following manner.

$C_\mu{}^\nu$ on one hand and $N_{(\mu\nu)}{}^{(\alpha\beta)}$, defined in (A 1), on the other

$$\mathring{g}_{(\mu\nu)} = N_{(\mu\nu)}{}^{(\alpha\beta)}\, g_{(\alpha\beta)} \tag{A1}$$

are two representations of the same group. Therefore the determinant of the $N_{(\mu\nu)}{}^{(\alpha\beta)}$ matrix must be a function, $f(x)$, of the determinant of the $C_\mu{}^\nu$ matrix, which satisfies

$$f(xy) = f(x)\cdot f(y), \quad (x,y>0). \tag{A2}$$

A priori we do not know if $f(x)$ is single valued. Strictly speaking, (A2) should therefore be interpreted as meaning, "The set of numbers $f(x)$ multiplied by the set of numbers $f(y)$ is the set of numbers $f(xy)$." This is true because

$$f(x)\cdot f(y) \in f(xy) \tag{A3}$$

$$1 \in f(x)\cdot f\left(\frac{1}{x}\right). \tag{A4}$$

Hence, we have

$$\begin{aligned} f(xy) = 1\cdot f(xy) &\in f(x)\cdot f\left(\frac{1}{x}\right) f(xy) \\ &\in f(x)\cdot f(y) \end{aligned} \tag{A5}$$

and (A2) follows from (A3) and (A5). Now equation (A2) gives

$$f\left(x^{\frac{n}{m}}\right) = [f(x)]^{\frac{n}{m}} \tag{A6}$$

where n and m are integers, and due to continuity we have

$$f(x^\alpha) = [f(x)]^\alpha \tag{A7}$$

where α is any real number. Putting the set of numbers $f(e)$ equal to e^β, with β a certain set of complex numbers independent of x, we get from inserting $x=e$ in (A7)

$$f(x) = x^\beta. \tag{A8}$$

On page 281 it is shown that β consists of one single number, viz. -5.

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